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Surface and internal waves in a stratified layer of liquid and an analysis of the impedance boundary conditions[☆]

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Abstract

The surface and internal waves in a multilayered ideal liquid for specified displacements of the bottom are considered. The upper surface of the liquid is either free or displacements are specified on it. In the long-wave approximation, asymptotically accurate models of the waves propagating along the surface layers of an incompressible liquid are constructed both for intense stratification (the ratio of the densities of neighbouring layers are associated with a small parameter) and for weak stratification. The important case of this problem of the dynamic contact of rigid bodies through a layer of incompressible or compressible liquid is investigated. High-order impedance boundary conditions are constructed and the results of testing them using the exact solutions are presented.

To analyse surface waves in a single layer, a method based on expansion in a series in powers of a small parameter, which is the ratio of the layer thickness to the wavelength, was previously used.[1] Below we investigate the general case of an arbitrary number of layers, based on an asymptotic analysis of the three-dimensional problem, taking into account strong or weak stratification of the layers. The boundary conditions on the upper surface of the liquid correspond either to a specified pressure or the condition that the liquid should be in contact with a rigid covering. The results of a similar analysis of the dynamic contact of rigid bodies through a layer of incompressible or compressible liquid enable one to construct the so-called high-order *impedance* boundary conditions and enable the dimension of the problem to be reduced over a reasonable frequency range without loss of accuracy. Unlike existing results,[2–4] the relations derived below can be used up to practically the first thickness resonance layer. In such applied problems as the sounding of elastic bodies with ultrasonic transducers through a thin layer of lubricant, this is of considerable importance, since it enables one to change to a problem of lower dimension by combining the long-wave approach to a thin layer with a consideration of short waves with respect to thick elastic solids. Here the actual frequencies can be extremely high (for millimeter thicknesses of the layer, up to tens of hundreds of MHz). The asymptotically accurate model obtained can also be used in problems of calculating the natural waves and spectra of the oscillations of thick multilayered bodies (half-spaces, plates or extremely shallow shells), in which contact between the elastic or viscoelastic layers is achieved via a liquid interface. Problems with initial conditions are not considered.

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1. Weak stratification

Suppose we have N infinite horizontal layers of immiscible ideal incompressible liquids, each of which occupies the region $-\infty < X_1, X_2 < \infty$, $Z_j \le Z \le Z_{j+1}$ ($j = 1, 2, \ldots$, N) and has a thickness $H_j = Z_{j+1} - Z$. The $Z = X_3$ axis is directed opposite to the force of gravity.

We will denote by $\mathbf{U}_j = (U_1, U_2, U_3)^j$ the displacement vector of points of the liquids in the j-th layer and Ψ_j are the displacement potentials, i.e. $U_3^j = \partial_{X_3}, \Psi_j$, $U_\alpha^j = \partial_{X_\alpha}\Psi_j$. The densities of the layers $\bar{\rho}_j$ satisfy the inequalities $\bar{\rho}_1 > \bar{\rho}_2 > \ldots > \bar{\rho}_N$. Correspondingly, $P_j = -\bar{\rho}_j \partial_T^2 \Psi_j$ is the pressure in the layer (T is the time). The vertical displacement U_3 when $Z = Z_j$ will be denoted by ζ_j. Henceforth, as usual, we will only consider the dynamic components of the pressure. The potentials Ψ_j satisfy Laplace's equations

$$(\partial_{X_1}^2 + \partial_{X_2}^2 + \partial_{X_3}^2)\Psi_j = 0, \quad j = 1, 2, \ldots, N \tag{1.1}$$

and the conditions on the interfaces of the layers $Z = Z_{j+1}$($j = 1, 2, \ldots N - 1$), i.e. the conditions of continuity of the vertical displacements

$$\zeta_{j+1} = \partial_{X_3}\Psi_{j+1} = \partial_{X_3}\Psi_j \tag{1.2}$$

and the conditions for the pressures to be matched (taking into account the excess gravitational component of the pressure)

$$-\bar{\rho}_{j+1}(\partial_T^2\Psi_{j+1} + \zeta_{j+1}) = -\bar{\rho}_j(\partial_T^2\Psi_j + \zeta_{j+1}) \tag{1.3}$$

On the bottom surface $Z^- \equiv Z_1$ the vertical displacement $\zeta^- \equiv \zeta_1$ will be assumed to be specified, i.e.

$$\partial_{X_3}\Psi_1 = \zeta_1 = \zeta^- \tag{1.4}$$

and for the first layer the function ζ_2 will be unknown. Similarly, numbering from the bottom to the top, together with the equation for the j-th layer, we will consider the new boundary $Z = Z_{j+1}$ and the function ζ_{j+1}.

The boundary conditions (1.2)–(1.4) must also be supplemented by the condition on the upper surface $Z^+ \equiv Z_{N+1}$. This condition can serve as the condition on the free surface, i.e. the internal pressure on the surface $Z = Z^+$ must be equated to the sum of the excess gravitational component of the pressure and the excess atmospheric pressure P_0^+ (if it exists)

$$-\bar{\rho}_N\partial_T^2\Psi_N = \bar{\rho}_N g\zeta_{N+1} + P_0^+ \tag{1.5}$$

For specified vertical displacements on the upper surface $Z = Z^+$ we obtain another version

$$\partial_{X_3}\Psi_N = \zeta_{N+1} = \zeta^+ \tag{1.6}$$

Eq. (1.1) and boundary conditions (1.2)–(1.5) (or (1.2)–(1.4), (1.6)) correspond to the exact formulation of the problem of the wave propagation in the multilayered medium described under the action of a perturbation specified on the upper and lower surfaces. The law of variation with time can be chosen to be harmonic – both for monochromatic waves and in the harmonic-synthesis problem.

We will now convert the scales. Suppose $H = H_1 + H_2 + \ldots + H_N$ is the total depth of the liquid layer and L is the characteristic scale of variability of the quantities in a longitudinal direction (the minimum wavelength). We will introduce the small parameter $\varepsilon = H/L \ll 1$ and change to the following dimensionless quantities

$$\rho_j = \frac{\bar{\rho}_j}{\bar{\rho}_0}, \ t = \frac{T}{T_0}, \ z = \frac{Z}{H}, \ x_\alpha = \frac{X_\alpha}{L}, \ \alpha = 1, 2, \ \mathbf{u} = \frac{\mathbf{U}}{H}, \ \psi = \frac{\Psi}{H^2}, \ \zeta^\pm = \frac{\zeta^\pm}{L}, \ p^+ = \frac{P^+}{\bar{\rho}_N C_0^2}$$

$$T_0 = LC_0^{-1}\varepsilon^{1-a/2}, \quad C_0 = \sqrt{gH}$$

where $\bar{\rho}_0$ is a certain characteristic density (for example, the maximum of the densities of the liquid layers), and the parameter a, to be determined, represents the "dynamicity" of the process.

We obtain the following relations for the fundamental operators and displacements

$$\partial_Z = H^{-1}\partial_z,\ \partial_{X_\alpha} = L^{-1}\partial_\alpha,\ \partial_T^2 = \varepsilon^a g H^{-1}\partial_t^2,\ \partial_{X_1}^2 + \partial_{X_2}^2 + \partial_{X_3}^2 = H^{-2}(\partial_z^2 + \varepsilon^2\Delta),\ \Delta \equiv \partial_1^2 + \partial_2^2$$

$$u_\alpha = \varepsilon\partial_\alpha\psi,\quad u_3 = \partial_z\psi$$

These formulae, in particular, clearly show the smallness of the derivatives in the longitudinal direction compared with the derivative in the transverse direction. Their ratio gives a quantity of the order of ε. Hence, we will assume that the derivatives with respect to the dimensionless coordinates ∂_α, ∂_z correspond to the variability of functions of one order ($O(1)$, $\varepsilon \to 0$).

We will represent the potentials, displacements and pressures in the layers in the form of asymptotic series (the subscript j of a layer will be omitted in obvious cases)

$$\Psi = H^2\varepsilon^\lambda(\psi^0 + \varepsilon\psi^1 + \dots)$$

$$U_\alpha = H\varepsilon^{\lambda+1}(u_\alpha^0 + \varepsilon u_\alpha^1 + \dots),\ U_3 = H\varepsilon^\lambda(u_3^0 + \varepsilon u_3^1 + \dots),\ P = \bar{\rho}_0 C_0^2\varepsilon^{\lambda+a}(p^0 + \varepsilon p^1 + \dots)$$

As it applies to the wave-propagation problems formulated above, we will derive a simplified description of these with the same relative error (with order ε of the omitted terms with respect to the retained terms) for all the physical quantities.

For the terms of the series, relations (1.1)–(1.6) take the following recurrent form

$$\partial_z^2\psi^{l,j} = -\Delta\psi^{l-2,j},\quad j = 1, 2, \dots, N \tag{1.7}$$

$$z = z_{j+1}\colon \zeta_{j+1}^l = \partial_z\psi^{l,j+1} = \partial_z\psi^{l,j},\ \rho_{j+1}\partial_t^2\psi^{l-a,j+1} - \rho_j\partial_t^2\psi^{l-a,j} = (\rho_j - \rho_{j+1})\zeta_{j+1}^l,$$
$$j = 1, 2, \dots, N-1 \tag{1.8}$$

$$z = z^-\colon \partial_z\psi^{l,1} = \delta_{-1}^{\lambda+l}\zeta^- \tag{1.9}$$

$$z = z^+\colon \zeta_{N+1}^l \equiv \partial_z\psi^{l,N} = -\partial_t^2\psi^{l-a,N} - p_0^+\delta_0^{\lambda+l} \tag{1.10}$$

$$z = z^+\colon \partial_z\psi^{l,N} = \delta_{-1}^{\lambda+l}\zeta^+ \tag{1.11}$$

where δ_q^p is the Kronecker delta.

We will first consider the case of a *free* upper surface of the liquid, including the case of a pressure P_0^+, specified on the surface. Relations (1.7)–(1.10) must be satisfied for the required terms of the asymptotic series. We will assume that $l = 0,1$. From Eq. (1.7) we then obtain $\psi^{l,j} = \psi_0^{l,j} + z\psi_1^{l,j}$, where the functions $\psi_0^{l,j}$ and $\psi_1^{l,j}$ are independent of z. Conditions (1.8)–(1.10) give

$$\zeta_{j+1}^l = \psi_1^{l,j+1} = \psi_1^{l,j} = \delta_{-1}^{\lambda+l}\zeta^-,\quad (\rho_j - \rho_{j+1})\psi_1^{l,j} = 0 \text{ (for } a > 0),\quad \psi_1^{l,N} = 0$$

This group of equalities will not be contradictory if $\lambda \neq -1, -2; a > 1$ and $\psi_1^{l,j} = 0$ (i.e. $\psi^{l,j} = \psi_0^{l,j}(\mathbf{x}, t), \mathbf{x} = (x_1, x_2)$).

From the permissible values we choose the first value $\lambda = -3$, which enables us to obtain an interesting relation at the boundary, and the least possible time scale $a = 2$, and we to consider the next two terms of the asymptotic series with number $l + 2 = 2,3$. Eq. (1.7) leads to the equality

$$\psi^{l+2,j} = \psi_0^{l+2,j} + z\psi_1^{l+2,j} - \frac{z^2}{2}\Delta\psi_0^{l,j} \tag{1.12}$$

where each function with a subscript is independent of z. From conditions (1.8)–(1.10) we obtain ($j = 1, 2, \dots, N-1$)

$$z = z_{j+1}\colon \zeta_{j+1}^{l+2} = \psi_1^{l+2,j+1} - z_{j+1}\Delta\psi_0^{l,j+1} = \psi_1^{l+2,j} - z_{j+1}\Delta\psi_0^{l,j} \tag{1.13}$$

$$z = z_{j+1}\colon \rho_{j+1}\partial_t^2\psi^{l,j+1} - \rho_j\partial_t^2\psi^{l,j} = (\rho_j - \rho_{j+1})\zeta_{j+1}^{l+2} \tag{1.14}$$

$$\zeta_{N+1}^{l+2} = \psi_1^{l+2,N} - z_{N+1}\Delta\psi_0^{l,N} = -\partial_t^2\psi^{l,N} - p^+\delta_0^{\lambda+l}, \quad \psi_1^{l+2,1} - z_1\Delta\psi_0^{l,1} = \delta_{-1}^{\lambda+l+2}\zeta^- \tag{1.15}$$

Hence, after algebraic reduction, we obtain the equations

$$\zeta_{j+1}^{l+2} - \zeta_j^{l+2} = -h_j\Delta\psi_0^{l,j}, \quad \zeta_1^{l+2} = \delta_{-1}^{l-1}\zeta^- \tag{1.16}$$

$$\rho_j\partial_t^2\psi_0^{l,j} = -\rho_j\zeta_{j+1}^{l+2} - \sum\nolimits_{j+1}\rho_k(\zeta_{k+1}^{l+2} - \zeta_k^{l+2}), \quad \sum\nolimits_{j+1} \equiv \sum_{j+1}^{N} \tag{1.17}$$

differentiating which with respect to the coordinates and time, we arrive at a system of equations for the functions of vertical displacement of the interfaces

$$\partial_t^2\zeta_j^{l+2} = -h_{j-1}\sum\nolimits_{j+1}\frac{\rho_k - \rho_{k-1}}{\rho_{j-1}}\Delta\zeta_j^{l+2} + h_{j-1}\frac{\rho_N}{\rho_{j-1}}\Delta\zeta_{N+1}^{l+2} + \partial_t^2\zeta_{j-1}^{l+2}, \quad j = 2, 3, \ldots, N$$

supplemented by the second equality of (1.16).

We will give the final dimensional form of the principal terms of the series for the required potentials and displacements

$$\Psi(\mathbf{x}, t) = L^2(\Psi^0 + \varepsilon\Psi^1), \quad U_\alpha = L(u_\alpha^0 + \varepsilon u_\alpha^1), \quad U_3 = H(u_3^0 + \varepsilon u_3^1) \tag{1.18}$$

The vertical displacements of points of the liquid in the j-th layer and the excess pressure on the interfaces of the layers can be expressed in terms of the dimensional vertical displacements at the layer interfaces and the potentials as follows:

$$U_3^j = \zeta_{j+1} + \frac{Z - Z_{j+1}}{H_j}(\zeta_{j+1} - \zeta_j), \quad U_\alpha^j = \partial_{X_\alpha}\Psi_j \tag{1.19}$$

$$Z = Z_{j+1}\colon P_{j+1} = g\{\bar\rho_j\zeta_{j+1} + \sum\nolimits_{j+1}\bar\rho_k(\zeta_{k+1} - \zeta_k)\} + P_0^+ \tag{1.20}$$

The potentials Ψ_j and the vertical displacements ζ_j satisfy the equations

$$\begin{gathered}\partial_T^2\zeta_j = a_{j-1}^2\left\{\sum\nolimits_j\frac{\bar\rho_{k-1} - \bar\rho_k}{\bar\rho_{j-1}}\Delta_X\zeta_k + \frac{\bar\rho_N}{\bar\rho_{j-1}}\Delta_X\zeta_{N+1} + \frac{P_0^+}{g\bar\rho_{j-1}}\right\} + \partial_T^2\zeta_{j-1}, \quad \zeta_1 = \zeta^- \\ \partial_T^2\Psi_j = -g\zeta_{j+1} - \sum\nolimits_j\frac{\bar\rho_k}{\bar\rho_j}a_k^2\Delta_X\Psi_k \ (a_j^2 \equiv gH_j, \ \Delta_X \equiv \partial_{X_1}^2 + \partial_{X_2}^2)\end{gathered} \tag{1.21}$$

In the special cases $N=1$ and $N=2$, Eq. (1.21) are identical with the existing equations.[5–7]

We will now investigate the case when the vertical *displacements* are specified on the upper surface of the multilayered liquid. The terms of the asymptotic series for the required quantities must then satisfy equalities (1.7)–(1.9) and (1.11). Since only conditions (1.10) are replaced by (1.11), then from the remaining conditions for the terms of the series with numbers $l=0,1$ we obtain

$$\psi^{l,j} = \psi_0^{l,j}(\mathbf{x}, t), \quad \lambda \neq -1, -2, \quad a > 1$$

A similar permissible combination $\lambda = -3$, $a = 2$ remains, and for the following terms of the series with numbers $l+2 = 2,3$ from relations (1.7)–(1.9) and (1.11) we will have equalities (1.12), (1.14) and (1.16) and the equation

$$\zeta_{N+1}^{l+2} = \delta_{-1}^{l-1}\zeta^+$$

Differentiating all these equalities with respect to the coordinates and time and adding, we arrive at the relations

$$\zeta_j^{l+2} = \delta_{-1}^{l-1}\zeta^{+} + \sum_{k=j-1}^{N} h_k \Delta\psi_0^{l,k} = \delta_{-1}^{l-1}\zeta^{-} - \sum_{k=1}^{j-2} h_k \Delta\psi_0^{l,k}$$

$$\left\{\partial_t^2 + \frac{(\rho_j - \rho_{j-1})h_j h_{j-1}}{\rho_{j-1}h_j + \rho_j h_{j-1}}\right\}\zeta_j^{l+2} = \partial_t^2\left\{\frac{\rho_{j-1}h_j\zeta_{j-1}^{l+2} + \rho_j h_{j-1}\zeta_{j+1}^{l+2}}{\rho_{j-1}h_j + \rho_j h_{j-1}}\right\}$$

For the principal terms of series (1.18), the main relations in dimensional form are as follows:

$$\begin{gathered}\partial_T^2\zeta_j = b_j^2\Delta_X\zeta_j + d_j^2\partial_T^2\zeta_{j-1} + e_j^2\partial_T^2\zeta_{j+1}; \quad \zeta_1 = \zeta^{-}, \quad \zeta_{N+1} = \zeta^{+} \\ b_j^2 = g(\bar{\rho}_{j-1} - \bar{\rho}_j)H_j H_{j-1}/\chi_j, \ d_j^2 = \bar{\rho}_{j-1}H_j/\chi_j, \ e_j^2 = \bar{\rho}_j H_{j-1}/\chi_j, \ \chi_j \equiv \bar{\rho}_{j-1}H_j + \bar{\rho}_j H_{j-1}\end{gathered} \tag{1.22}$$

$$\zeta_{j+1} - \zeta_j = -H_j\Delta_X\Psi_j, \quad P_j - P_i = g\sum_{k=i+1}^{j}(\bar{\rho}_{k-1} - \bar{\rho}_k)\zeta_k \tag{1.23}$$

Relations (1.19) for the transverse and longitudinal displacements remain unchanged.

2. Strong stratification

We will now assume that the ratio of the densities in the liquid layers are of the order of the small parameter ε to a certain power, i.e.

$$\bar{\rho}_N : \bar{\rho}_{N-1} : \ldots : \bar{\rho}_2 : \bar{\rho}_1 \sim \varepsilon^{\beta_N} : \varepsilon^{\beta_N} : \ldots : \varepsilon^{\beta_2} : \varepsilon^{\beta_1}$$

$$\beta_N \geq \beta_{N-1} \geq \ldots \geq \beta_2 \geq \beta_1 \geq 0, \quad \delta_j \equiv \beta_{j-1} - \beta_j$$

We will divide the layers into sub-groups, so that inside each of them $\delta_j = 0$ for neighbouring layers. Assuming, to fix our ideas, that the quantities β_j are integers, we will introduce the notation N_m for the set of indices of these layers, $\beta_j = m$. Hence, a strongly stratified liquid will be defined by the collection of sets $N_0, N_1, N_2, \ldots$, in order of increasing indices of the layers from the heavy bottom layers to the lighter surface layers. Since, in the long-wave approximation, one usually assumes $\varepsilon \sim 10^{-2}$, it is natural to confine ourselves to considering a few actual values $m = 0,1,2$.[8,9]

The case of a liquid corresponding to the set N_0 has already been considered in Section 1; we will now consider combinations of different values of m. We will keep the procedure for scaling the coordinates and time unchanged. The form of the asymptotic series for the potentials and displacements is also retained, but for the pressures and densities we will have

$$P_j = C_0^2\varepsilon^{\lambda + a + \beta_j}(p^0 + \varepsilon p^1 + \ldots)_j, \quad \bar{\rho}_j = \bar{\rho}_{0j}\varepsilon^{\beta_j}, \quad \bar{\rho}_{0j} = O(1), \quad \varepsilon \to +0$$

The conditions for the pressures to be matched are correspondingly changed, and instead of the second condition of (1.8) we obtain the condition

$$-\rho_{0j}(\partial_t^2\psi^{l-a,j} + \zeta_{j+1}^{l}) = -\rho_{0j+1}(\partial_t^2\psi^{l-a-\delta_j,j+1} + \zeta_{j+1}^{l-\delta_j}) \tag{2.1}$$

where ρ_{0j} are dimensionless densities of the order of unity. Eq. (1.7), the first condition of (1.8) and conditions (1.9)–(1.11) remain in force.

We will first consider a liquid consisting of two groups of layers N_0, $N_2(N_0 : j = 1, 2, \ldots, N_0)$ and $N_2 : j = N_0 + 1, N_0 + 2, \ldots, N_0 + N_2; N = N_0 + N_2$, and we will construct its asymptotic model with a relative error $O(\varepsilon^2)$. It is obvious that $\delta_{N_0} = 2$, while the remaining $\delta_j = 0$, since adjacent layers will belong to one group. For the components of the asymptotic series with superscript $l = 0,1$ the calculations do not differ from those presented in Section 1. For the next terms $l+2=2,3$ with the same permissible values of $\lambda=-3$, $a=2$, a difference only occurs in the second condition (1.8) on the interface of groups N_0 and N_2, while condition (2.1) takes the form

$$\rho_{0N_0}(\partial_t^2\psi^{l,N_0}+\zeta_{N_0+1}^{l+2}) = 0 \tag{2.2}$$

Hence, we can speak of a separate description of the groups of layers. According to condition (2.2), the group of layers N_0 behaves as a liquid with a *free* boundary $Z=Z_{N_0+1}$ for a specified perturbation ζ^- on the bottom $Z=Z^-$, and the corresponding dimensional equations have the form (1.18)–(1.21) with $P^+=0$ for all $j\in N_0$. The group of layers N_2 takes the interface $Z=Z_{N_0+1}$ as its own "bottom" with specified perturbation ζ_{N_0+1} (obtained from the solution for the group of layers N_0). Depending on whether the condition on the upper surface $Z=Z^+$ is specified in the form of a pressure or a displacement, the behaviour of the group N_2 is described either by dimensional equations of the form (1.18)–(1.21), or by equations of the form (1.22), (1.23). Only the indices of the layers must be changed and counted from the interface $Z=Z_{N_0+1}$.

Obviously all these conclusions also remain the same for a liquid consisting of groups of layers N_0, N_m $(m>2)$ with the same accuracy $O(\varepsilon^2)$.

For a multilayered liquid of groups of layers N_0 and N_1 the relations presented above and the conclusions remain the same for terms of the asymptotic series $l=0,1$ and $l+2=2$. Hence, with an accuracy $O(\varepsilon)$ we can use a separate description of the groups of layers N_0 and N_1. If we take both terms of the series $l+2=2,3$ into account, condition (2.2) must be replaced by the relation

$$-\rho_{0N_0}(\partial_t^2\psi^{l,j}+\zeta_{N_0+1}^{l+2}) = -\rho_{0N_0+1}(\partial_t^2\psi^{l-1,N_0+1}+\zeta_{N_0+1}^{l+1}) \tag{2.3}$$

i.e. the second term of the asymptotic form for the group of layers N_0 is "perturbed" by the excess weight pressure of the group of layers N_1 of lower order. Relation (2.3) essentially denotes regrouping of the terms of the asymptotic series, and it is easy to verify that, with an error $O(\varepsilon^2)$, a liquid from the groups of layers N_0 and N_1 satisfies the same dimensional equations as the weakly stratified liquid from Section 1.

If we consider a more abstract combination of groups of layers of the form $N_0, N_1, \ldots, N_m$, then, with an accuracy $O(\varepsilon)$, their description may be separate. Here, for each group N_k, $k<m$ the lower interface is taken as its own "bottom", while the upper one is "freed" from the pressure. If between groups of layers $\delta_j=1$, then, with an accuracy $O(\varepsilon^2)$, we can use the equations presented in Section 1 for a connected description of a weakly stratified liquid. If, between certain neighbouring groups $\delta_j\geq 2$, the equations for the groups of layers of liquids situated above and below the common interface are completely separated like the combination of the group of layers N_1, and N_2.

3. Impedance boundary conditions

3.1. The case of an incompressible liquid

We will now consider a single layer of liquid, constrained by two plane solid surfaces. For convenience, in addition to the thickness of the layer, we will also introduce the half-thickness $h=H/2$. From relations (1.7), the first condition of (1.8) and conditions (1.9) and (1.11), we obtain the following recurrence formulae

$$\begin{gathered}
\psi^{l+2n} = \sum_{k=0}^{2n}\frac{z^k}{k!}\psi_k^{l+2n}, \quad u_z^{l+2n} = \sum_{k=1}^{2n}\frac{z^{k-1}}{(k-1)!}\psi_k^{l+2n}; \quad l=0,1; \quad n=0,1,\ldots \\
p^{l+2n} = -\partial_t^2\psi^{l+2n}, \quad \psi_{k+2}^{l+2n} = -\Delta\psi_k^{l+2n-2} \\
\psi_1^{l+2n} = \delta_+^{l+2n}-\delta_-^{l+2n}, \quad \delta_-^{l+2n} = \frac{1}{2}\sum_{k=2}^{2n}\frac{1}{(k-1)!}\psi_k^{l+2n}; \quad \delta_\pm\equiv\frac{\zeta^+\pm\zeta^-}{2}
\end{gathered} \tag{3.1}$$

For the same values of the parameters λ and a as in Section 1, like the quantities $\delta_\pm$ we will introduce into consideration the half-sum and half-difference of the pressures on the upper and lower surfaces of the layer $P=P^\pm$.

Following the previous calculations, for the indices $l = 0,1$ we obtain

$$\zeta^l = \psi_1^l = 0, \quad \psi_0^l \neq 0, \quad p^l = -\partial_t^2 \psi_0^l(\mathbf{x}, t) = p^{l,\pm}$$

$$\frac{1}{2}(p^{l,+} - p^{l,-}) = 0, \quad \frac{1}{2}(p^{l,+} + p^{l,-}) = -\partial_t^2 \psi_0^l(\mathbf{x}, t),$$

For the next pair of indices $l + 2 = 2,3$, the relations take the form

$$\zeta^{l+2} = \psi_1^{l+2} - z\Delta\psi_0^l, \quad -\frac{1}{2}\Delta\psi_0^l = \delta_-^{l+2}, \quad -\frac{1}{2}\partial_t^2\Delta\psi_0^l = \frac{1}{2}\Delta(p^{l,+} + p^{l,-}) = \partial_t^2\delta_-^{l+2}$$

with the two final equations

$$\frac{1}{2}(p^{l,+} - p^{l,-}) = 0, \quad \frac{1}{2}\Delta(p^{l,+} + p^{l,-}) = \partial_t^2\delta_-^{l+2} \tag{3.2}$$

Considering the indices $n = 2, 3, 4$ successively in relations (3.1) and changing to dimensional formulae, we obtain, like (3.2), the following equations for the leading indices $n = 4$ with a relative error $O(\varepsilon^{10})$

$$\begin{aligned} &P^+ - P^- = -h\bar{\rho}\partial_T^2 D^-(U_z^+ + U_z^-), \quad h\Delta_X(P^+ + P^-) = \partial_T^2 D^+(U_z^+ - U_z^-) \\ &D^- \equiv 0 + 1 + \frac{1}{3}h^2\Delta_X + \frac{2}{15}h^4\Delta_X^2 + \frac{17}{315}h^6\Delta_X^3 \\ &D^+ \equiv 1 - \frac{1}{3}h^2\Delta_X - \frac{1}{45}h^4\Delta_X^2 - \frac{2}{945}h^6\Delta_X^3 - \frac{1343}{38102400}h^8\Delta_X^4 \end{aligned} \tag{3.3}$$

The relations of lower orders are obtained by dropping the last terms in the operators $D^{\pm}$ in Eq. (3.3). In the limiting case when $h \to +0$ we have $P^+ = P^-$, $U_z^+ = U_z^-$.

Suppose now that the layer of incompressible liquid is situated between two elastic bodies, each of which can be, for example, a thick plate, an extremely shallow shell or a half-space of isotropic or anisotropic linearly elastic or viscoelastic material. All the quantities relating to the elastic bodies will be denoted by a "plus" or "minus" superscript. We must add to conditions (1.4) and (1.6) on the interphase surfaces $Z = Z^{\pm}$ the impermeability conditions (here $U_z^{\pm}$ are the normal displacements of the elastic medium)

$$Z = Z^{\pm}: U_z^{\pm} = \zeta^{\pm} \tag{3.4}$$

The normal stresses must satisfy the pressure-balance condition

$$Z = Z^{\pm}: P^{\pm} = -\sigma_{zz}^{\pm} + (\bar{\rho}_{\pm} + \bar{\rho})gU_z^{\pm}; \quad \sigma_{\alpha z}^{\pm} = 0, \quad \alpha = 1, 2 \tag{3.5}$$

Scaling of the quantities and asymptotic integration in the liquid layer were carried out using the method described above. In elastic bodies, scaling of the coordinates X_1, X_2 and Z is only carried out for one linear dimension L. This means that for corresponding terms of the asymptotic series in thick bodies, which satisfy relations (3.4) and (3.5), the usual equations of the dynamic theory of elasticity or viscoelasticity are satisfied, since such scaling does not lead to mixing of the different indices in the relations if the time scale is no less than the characteristic time of the elastic processes. As a result we arrive at the same Eq. (3.3), where the pressures are expressed in terms of the normal stresses, defined by the first formula of (3.5). The quantity of the order λ is not specified in this case. The final impedance boundary conditions of the tenth order on the interphase surfaces $Z = Z^{\pm}$ take the form

$$\begin{aligned} &-\sigma_{zz}^+ + (\bar{\rho}_+ + \bar{\rho})gU_z^+ + \sigma_{zz}^- - (\bar{\rho}_- + \bar{\rho})gU_z^- = -h\bar{\rho}\partial_T^2 D^-(U_z^+ + U_z^-) \\ &h\Delta_X\{-\sigma_{zz}^+ + (\bar{\rho}_+ + \bar{\rho})gU_z^+ - \sigma_{zz}^- + (\bar{\rho}_- + \bar{\rho})gU_z^-\} = \partial_T^2 D^+(U_z^+ - U_z^-) \end{aligned} \tag{3.6}$$

The conditions of the lower orders, as above, are obtained by dropping the last terms in the operators $D^{\pm}$ on the right-hand sides of Eq. (3.6).

The impedance boundary conditions presented above were obtained for the case when the ratios of the densities of the elastic bodies to the density of the liquid are quantities $O(1)$ as $\varepsilon \to 0$. It is physically obvious that the case of a light liquid $\bar{\rho} \sim \varepsilon\bar{\rho}_{\pm}$ will also be described by Eq. (3.6). The case of a heavy liquid between extremely pliable light bodies is not considered here.

We will illustrate the range of applicability of the impedance boundary conditions obtained using the test example of the calculation of the characteristics of partial waves. This test is completely acceptable for estimating the accuracy of the asymptotic model, since partial waves arise in a natural way when considering any spectral problem. We will consider the propagation of waves in half-spaces of aluminium (density 2700 kg/m^3 and velocities $C_p = 6320$ and $C_S = 3080$ m/s) and/or polystyrene (density 1060 kg/m^3, $C_p = 2350$ and $C_S = 1150$ m/s) through a layer of water (density 1060 kg/m^3), considered initially as an incompressible liquid. For the case of the incidence of a time-harmonic P-wave or S-wave from the upper medium onto the interface with the layer of liquid we will determine the complex amplitudes M for the P-wave and S-wave, reflected in the upper medium, and waves passing into the lower medium, referred to the amplitude of the incident wave M^{in}, and also the root mean square relative error

$$e = \left\{\frac{1}{4}\sum\left|\frac{M^{as} - M^{ex}}{M^{ex}}\right|^2\right\}^{1/2}$$

where M^{ex} is the complex amplitude of the corresponding wave, obtained from the exact solution of the problem of hydroelastic contact and M^{as} is the amplitude obtained from the solution of the approximate problem, where the effect of the liquid is taken into account by means of the impedance boundary conditions (3.6), and the summation is carried out over all types of waves in the elastic media.

The results of calculations of the maximum values of H/L, for which the error of the impedance boundary conditions $e \leq 0.01$ for different angles of incidence of the P- and S-waves on the interface of the elastic body with the incompressible liquid, are presented in the upper rows of Table 1. We used a wavelength $L = L^{+} \equiv 2\pi C_S^{+}/\omega$ as a wave scale (ω is the angular frequency). The column numbers n correspond to an asymptotic error $O(\varepsilon^{2n})$ in impedance boundary conditions of the form (3.6). As can be seen, the range of applicability of the impedance boundary conditions is extremely wide, up to cases when the parameter of the asymptotic expansion $\varepsilon = h/L$ or the wave number $kh = 2\pi\varepsilon = \pi H/L$ cannot be assumed to be small. Note that although the scaling was carried out for a velocity of a gravitational wave in a liquid, the final result also holds in the scale of velocities of elastic waves. Note also the considerable difference in the effectiveness of the impedance boundary conditions of different asymptotic orders.

3.2. *The case of a compressible liquid*

We will briefly describe only the main differences from the previous case. Instead of Eq. (1.1) for the potential of the displacements we will use the equation

$$\{C^{-2}\partial_T^2 - \partial_Z^2 - \Delta_X\}\Psi = 0$$

where C is the velocity of sound in the liquid. The time scale will be specified by the formula $T_0 = \varepsilon^a L/C$. The maximum permissible value $a = 0$ follows from the subsequent analysis (similar to that in Sections 1 and 3). The remaining calculations are similar to those above if the dimensionless operator $\dot{\Delta}$ is replaced by $\Delta - \partial_t^2$. In the dimensional relations (3.3) and in the impedance boundary conditions (3.6) we must replace Δ_X by $\Delta_X - C^{-2}\partial_T^2$. Note that, for the majority of actual physical materials, the "weight" addition to the stresses in formulae (3.6) is small and $p^{\pm} \approx -\sigma_{zz}^{\pm}$. Note also that the case of a heavy liquid or a liquid in which the velocity of sound is much greater than the wave propagation velocities in elastic media is not considered here.

Table 1

n	1	2	3	4	5
Aluminium-water-aluminium	0.015	0.4	0.65	0.8	0.9
	0.015	0.12	0.2	0.26	0.31
Polystyrene-water-aluminium	0.02	0.6	1.1	1.3	1.4
	0.02	0.13	0.22	0.28	0.32

For illustration we will again consider the contact between the same media, where now water is the compressible medium with a velocity of sound $C = 1400$ m/s. The ratio $H/L = 1/2$ ($L = 2\pi C/\omega$) corresponds to the first quasi-resonance frequency of the liquid layer. The results of calculations are given in the lowest rows of the table.

It can be seen that for impedance boundary conditions (3.6) with $n = 5$ the relative error $e \leq 0.01$ if $H/L \leq 0.3$ (for $H/L = 0.4$ it amounts to only 0.05). A number of numerical calculations for other materials, which are not given here, confirm this conclusions.

It is significant that, unlike the results obtained previously[2–4] for impedance boundary conditions of the lower order, the results presented here cover practically the whole frequency range up to the first quasi-resonance. The main difference is the occurrence on the right-hand side of impedance boundary conditions (3.6) of iterations of the wave operator $\Delta_X - C^{-2}\partial_T^2$, which occurs when $n \geq 2$, which also explains the sharp increase in the accuracy of the model and the approximation of the frequency of the first quasi-resonance in the liquid layer.

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